# Lecture 29

## 11.9 - Representing Functions as Power Series

### Starting Point

From our knowledge of geometric series, we have a power series representation of $g(x) = \frac{1}{1-x}$ as:

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \text{ for } |x|<1 \Leftrightarrow -1<x<1.$$

Using this as our model, we can represent more functions as power series:

Ex: Represent the following functions as power series and find the interval of convergence:

ⓐ $$f(x) = \frac{2}{4-x} = \frac{1}{4}\left(\frac{2}{1-\frac{x}{4}}\right) = \frac{\frac{1}{2}}{1-\frac{x}{4}} = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{2n+1}}$$

for $\left|\frac{x}{4}\right|<1 \Leftrightarrow |x|<4 \Leftrightarrow -4<x<4$

different centers!

—or—

$$f(x) = \frac{2}{4-x} = \frac{2}{1+3-x} = \frac{2}{1-(x-3)} = \sum_{n=0}^{\infty} 2(x-3)^n$$

for $|x-3|<1 \Leftrightarrow -1<x-3<1 \Leftrightarrow 2<x<4.$

(b) $g(x) = \dfrac{x^2}{1+x} = \dfrac{x^2}{1-(-x)} = \sum_{n=0}^{\infty} x^2(-x)^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$

valid for $|-x|<1 \iff -1<x<1$

(c) $h(x) = \dfrac{-3}{x^2+2x} = \dfrac{-3}{x^2+2x+1-1} = \dfrac{-3}{(x+1)^2-1} = \dfrac{3}{1-(x+1)^2}$

$= \sum_{n=0}^{\infty} 3(x+1)^{2n}$

valid for $|(x+1)^2|<1 \iff |x+1|<1 \iff -1<x+1<1$

$\iff -2<x<0$

⚠ Be careful on this one: we don't want to factor out an $x$ or $x^2$ from the bottom because it creates negative powers of $x$ in the series!

# Differentiation & Integration of Power Series

Just as with normal polynomials, we can differentiate and integrate a power series term-by-term.

Theorem: If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R$ then the function $f$ defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ with

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} c_n n(x-a)^{n-1}$$

and is also integrable with

$$\int f(x)\,dx = c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots + C = \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} + C$$

The radius of convergence for both of these series is $R$. Regardless of the convergence at the endpoints for $f(x)$, after integrating or differentiating, convergence at endpoints should be checked again.

Ex: Find a power series representation of

$$f(x)=\frac{1}{(x+1)^2}$$

Notice: $f(x)=g'(x)$ where $g(x)=\frac{-1}{x+1}=\frac{-1}{1-(-x)}$

$$g(x)=\frac{-1}{1-(-x)}=\sum_{n=0}^{\infty}(-1)(-x)^n=\sum_{n=0}^{\infty}(-1)^{n+1}x^n \quad \text{for } |x|<1$$

$$f(x)=g'(x)=\left(\sum_{n=0}^{\infty}(-1)^{n+1}x^n\right)'=\sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1} \quad \text{for } |x|<1$$

(when $x=1$: $\sum_{n=1}^{\infty}(-1)^{n+1}n$ diverges by the divergence test | when $x=-1$: $\sum_{n=1}^{\infty}(-1)^{n+1}n(-1)^{n-1}=\sum_{n=1}^{\infty}n$ diverges by the divergence test)

Ex: Find a power series representation of $g(x)=\arctan(x)$.

We know $\arctan x=\int\frac{1}{1+x^2}dx$

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n} \quad \text{for } |x^2|<1 \Leftrightarrow |x|<1$$

So, $$\arctan x=\int\frac{1}{1+x^2}dx=\int\sum_{n=0}^{\infty}(-1)^n x^{2n}dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}+C$$

for $|-x^2|<1 \Leftrightarrow |x|<1$

& $$0=\arctan 0=\sum_{n=0}^{\infty}\frac{(-1)^n 0^{2n+1}}{2n+1}+C=C$$

Thus $$\arctan x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \quad \text{for } |x|<1$$

(Check this diverges for $x=\pm 1$)

Ex: Find the sum of the series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n+1}(2n+1)}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n+1}(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$$

since $\left|\frac{1}{\sqrt{3}}\right| < 1$ $\quad = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$